The University of Nottingham

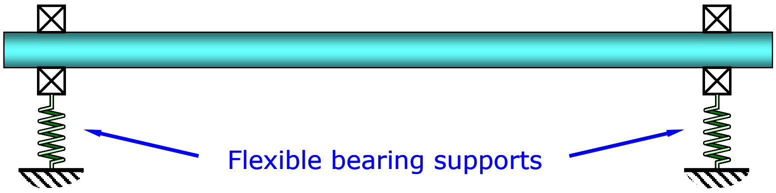
SHAFT WHIRL, CRITICAL SPEEDS & BEAM VIBRATION
MODULE: MMME2046 DYNAMICS & CONTROL

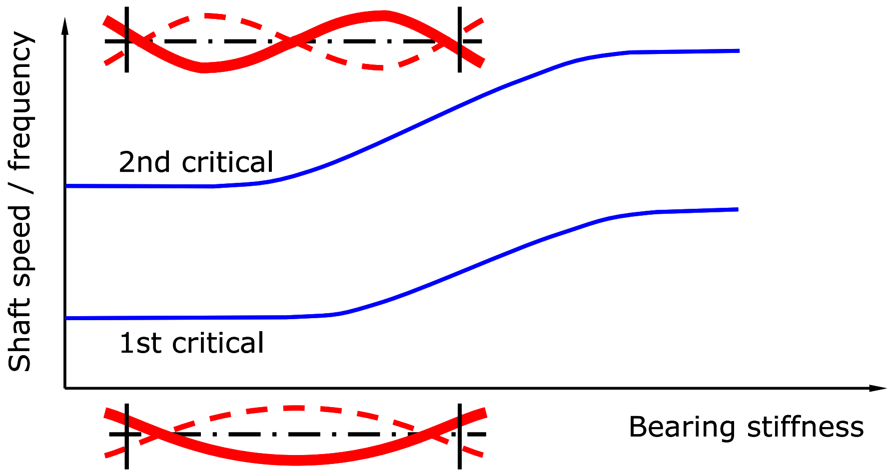
Shaft Whirl and Critical Speeds

Shaft whirl is a potentially destructive, self-sustaining flexural vibration observed in rotating shafts. It occurs if the rotational frequency of the shaft coincides with a resonant frequency for flexural vibration. These shaft speeds are called *critical speeds*. The analysis that follows shows that shafts have an infinite number of flexural resonant frequencies, which means that they have an infinite number of critical speeds.

A given shaft will be designed to operate with some maximum speed. Ideally, if this maximum design speed is less than the lowest critical speed, whirl will not be a problem. Unfortunately, this is not always possible and it is vital to be able to calculate what the critical speeds will be. We will do this by modelling the shaft as a "beam" with a circular cross section.

Short case study – High speed drive shaft



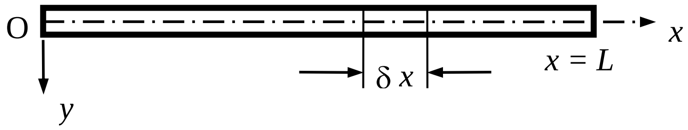


Make your own notes on this

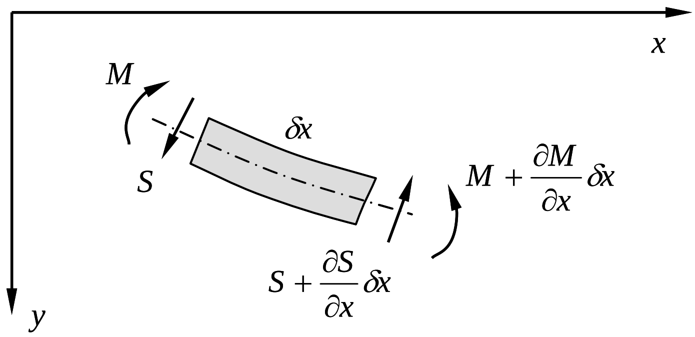
Other Beam-like Structures

Apart from shafts, many structures exhibit beam-like vibration behaviour. Examples include aircraft wings, helicopter rotor blades and tall chimneys (all of which vibrate in response to aerodynamic buffeting) and tall buildings that vibrate significantly during earthquakes. While these are more complex than uniform beams, they exhibit many of the same characteristics. This section of the module will therefore provide good insight into this behaviour.

Analysis of the Flexural Vibration of Uniform Beams



Unlike previous cases, a beam does not consist of discrete masses connected by massless springs. Both mass and stiffness are distributed along the length. A different approach is required and we start by considering an infinitesimal element of the beam of length δx.



Bending moment - curvature relationship is $M = -E\,I\,\dfrac{\partial^2 y}{\partial x^2}$ (1)

Equation for vertical motion : *Downwards is positive*

$$S - \left(S + \frac{\partial S}{\partial x}\,\delta x \right) = \left(\rho\, A\, \delta x \right) \frac{\partial^2 y}{\partial t^2}$$

$$\therefore \frac{\partial S}{\partial x} = -\rho\, A\, \frac{\partial^2 y}{\partial t^2} \qquad (2)$$

If we neglect the rotational moment of inertia of the element[1], the equation for rotational motion about an axis through the centre of mass of the element is

$$S\,\frac{\delta x}{2} + \left(S + \frac{\partial S}{\partial x}\,\delta x \right)\frac{\delta x}{2} - M + \left(M + \frac{\partial M}{\partial x}\,\delta x \right) = 0$$

$$\therefore S = -\frac{\partial M}{\partial x} \qquad (3)$$

Substituting for M from (1) into (3) and then for *S* in (2) we get

$$E\,I\,\frac{\partial^4 y}{\partial x^4} = -\rho\, A\, \frac{\partial^2 y}{\partial t^2} \qquad (4)$$

This is the governing differential equation for the free vibration of the beam.

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1 In addition to neglecting the ROTATIONAL INERTIA of the beam, the theory also neglects SHEAR DEFORMATION of planes of cross-section. Both assumptions tend to give an ***over-estimate*** of the natural frequencies of the beam. While this error is normally small for the first few modes, it increases progressively when higher frequencies are evaluated.

Equation (4) is a partial differential equation giving the deflection, y, which is a function of space x and time t. The objective in solving the equation will be to find the natural frequencies and the corresponding mode shapes.

For free vibration at a natural frequency, the displacement of any point on the beam in the y-direction will be sinusoidal, but the amplitude of the vibration will vary along the length. We can therefore use as a substitution,

$$y(x, t) = Y(x) \cos \omega t$$

Substituting into (4), we get

$$E I \frac{d^4 Y}{d x^4} \cos \omega t = \rho A \omega^2 Y(x) \cos \omega t$$

$$\therefore \quad \frac{d^4 Y}{d x^4} = \frac{\rho A \omega^2}{E I} Y(x)$$

For a uniform cross-section, A and I are constant and it is convenient to introduce the so-called ***wavenumber***, λ, defined by

$$\lambda^4 = \frac{\rho A \omega^2}{E I} \tag{5}$$

$$\text{to give} \qquad \frac{d^4 Y}{d x^4} = \lambda^4 Y(x)$$

$$\text{Take as solution} \qquad Y(x) = A\, e^{\alpha x}$$

$$\text{Thus,} \qquad \alpha^4 A e^{\alpha x} = \lambda^4 A e^{\alpha x}$$

$$\alpha^4 = \lambda^4$$

$$\text{so that} \qquad \alpha = \pm \lambda \quad \text{or} \quad \pm \mathbf{i} \lambda$$

The complete solution for $Y(x)$ is therefore

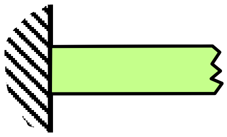
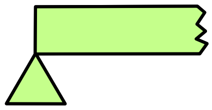
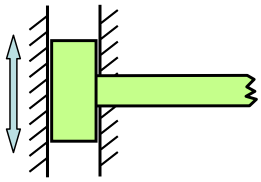
$$Y(x) = A_1 e^{\lambda x} + A_2 e^{-\lambda x} + A_3 e^{\mathbf{i} \lambda x} + A_4 e^{-\mathbf{i} \lambda x}$$

which may be rewritten to give the more convenient form,

$$Y(x) = C_1 \sin \lambda x + C_2 \cos \lambda x + C_3 \sinh \lambda x + C_4 \cosh \lambda x \tag{6}$$

This is a **general equation** giving the deflected shape of any beam of uniform cross-section. It is one of the equations given on the formula sheet. The constants C_1 - C_4 need to be determined from the boundary conditions at the ends of the beam. In this module we consider 4 basic types of support. The appropriate boundary conditions are given at the top of the next page.

Other types of boundary conditions are considered on page 9.

Descriptive terms	Diagrammatic	Boundary conditions
Built-in clamped encastré		$y = 0 \quad \dfrac{\partial y}{\partial x} = 0$
Simple support hinged pinned		$y = 0$ $M = 0 \;\therefore\; \dfrac{\partial^2 y}{\partial x^2} = 0$
Free		$M = 0 \;\therefore\; \dfrac{\partial^2 y}{\partial x^2} = 0$ $S = 0 \;\therefore\; \dfrac{\partial^3 y}{\partial x^3} = 0$
Massless slider		$\dfrac{\partial y}{\partial x} = 0$ $S = 0 \;\therefore\; \dfrac{\partial^3 y}{\partial x^3} = 0$

General Approach for Finding the Solutions for Particular Cases

1. Start by identifying the four boundary conditions. Use $y(x, t) = Y(x)\cos\omega t$, with equation (6) to express the boundary condition in terms of $Y(x)$ and its derivatives.

2. Since each of the boundary condition equations depends on C_1 - C_4, they can be assembled in the form

$$[Z]\{C\} = \{0\} \tag{7}$$

   where $\{C\}$ is a vector of the constants C_1 - C_4 and $[Z]$ is a coefficient matrix.

3. For a valid solution, $\det[Z] = 0$.

   This gives the **frequency equation** and its roots will give the natural frequencies of the beam.

4. When each root is substituted back into equation (7), the solution vector $\{C\}$ will define the corresponding **mode shape** when the values are put into equation (6).

Note: The help pages on the Moodle site have several resources related to this topic. These include a reminder of how to evaluate a 4x4 determinant and some Matlab programs that give animated examples of the mode shapes of beams.

Frequency equation for particular end conditions

Pinned-pinned	$\sin \lambda L = 0$
Clamped-clamped & free-free	$\cos \lambda L \cosh \lambda L - 1 = 0$
Clamped-pinned & free-pinned	$\tan \lambda L - \tanh \lambda L = 0$
Clamped-free	$\cos \lambda L \cosh \lambda L + 1 = 0$

Numerical values of roots, $\lambda_r L$, of frequency equations

r	1	2	3	4	5	>5
Pinned-pinned	π	2π	3π	4π	5π	$r\pi$
Clamped-clamped & free-free*	4.730	7.853	10.996	14.137	17.279	$\approx (r + 0.5)\pi$
Clamped-pinned & free-pinned	3.927	7.069	10.210	13.351	16.493	$\approx (r + 0.25)\pi$
Clamped-free	1.875	4.694	7.855	10.996	14.137	$\approx (r - 0.5)\pi$

* A free-free beam will also have 2 rigid body modes corresponding to $\lambda L = 0$.

Selecting the values of $\lambda_r L$ from the above table for the beam of interest, the natural frequencies can be found from equation (5). That is: $\omega_r = \dfrac{(\lambda_r L)^2}{L^2}\sqrt{\dfrac{EI}{\rho A}}$

Example 1 Simply-supported Beam

① The boundary conditions at $x = 0$ and at $x = L$ are

Since $y(x, t) = Y(x)\cos \omega t$, the boundary conditions become

From equation (6)

$$Y(x) = C_1 \sin \lambda x + C_2 \cos \lambda x + C_3 \sinh \lambda x + C_4 \cosh \lambda x$$

$$\frac{d^2Y}{dx^2} =$$

Note that $\dfrac{d}{d\theta}\sinh\theta = \cosh\theta$ and $\dfrac{d}{d\theta}\cosh\theta = \sinh\theta$

Look at the help page on the Moodle site if you don't know what the sinh and cosh functions look like.

Hence, at x = 0

and at x = L

② Assembling the four equations in matrix form;

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & -\lambda^2 & 0 & \lambda^2 \\ \sin\lambda L & \cos\lambda L & \sinh\lambda L & \cosh\lambda L \\ -\lambda^2\sin\lambda L & -\lambda^2\cos\lambda L & \lambda^2\sinh\lambda L & \lambda^2\cosh\lambda L \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{7}$$

③ This is the particular form of equation (7) for a simply-supported beam. Expanding the determinant of the coefficient matrix and equating to zero gives the **Frequency Equation.**

$$-4\lambda^4 \sin\lambda L \sinh\lambda L = 0$$

Q1. What are the roots of the equation?

Q2. Can λ = 0 ?

The frequency equation therefore reduces to

which has roots $\lambda_r L = r\pi$ for r = 1, 2, 3, …

From equation (5), the natural frequencies are $\omega_r = \left(\frac{r\pi}{L}\right)^2 \sqrt{\frac{EI}{\rho A}}$ for r = 1, 2, 3, ...

④ To find the corresponding mode shapes, substitute the roots into equation (7) and solve for the constants C_1 - C_4

Continue on additional sheets

Example 2 VIBRATION OF A CANTILEVER (CLAMPED-FREE) BEAM

Consider a cantilever that is clamped at x = 0 and free at $x = L$.

① The boundary conditions are:

$$\text{At} \quad x = 0\,, \quad y = 0 \ \text{and} \ \frac{\partial y}{\partial x} = 0$$

$$\text{At} \quad x = L\,, \quad M = 0 \quad \therefore \frac{\partial^2 y}{\partial x^2} = 0$$

$$\text{and} \quad S = 0 \quad \therefore \frac{\partial^3 y}{\partial x^3} = 0$$

Since $y\left(x\,,t\right) = Y\left(x\right)\cos\omega t$, the end conditions become

$$\text{At} \ \ x = 0\,, \quad Y = 0 \ \ \text{and} \ \ \frac{\mathrm{d}\,Y}{\mathrm{d}\,x} = 0$$

$$\text{At} \ \ x = L, \quad \frac{\mathrm{d}^2 Y}{\mathrm{d}\,x^2} = 0 \ \ \text{and} \ \ \frac{\mathrm{d}^3 Y}{\mathrm{d}\,x^3} = 0$$

② Substituting from equation (6) we get (in matrix form),

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ \lambda & 0 & \lambda & 0 \\ -\lambda^2 \sin\lambda L & -\lambda^2 \cos\lambda L & \lambda^2 \sinh\lambda L & \lambda^2 \cosh\lambda L \\ -\lambda^3 \cos\lambda L & \lambda^3 \sin\lambda L & \lambda^3 \cosh\lambda L & \lambda^3 \sinh\lambda L \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{7}$$

This is the particular version of equation (7) for a cantilever beam.

③ The **frequency equation** is given by setting the determinant of the coefficients of C_1 - C_4 to zero. After some manipulation (and noting that a cantilever has no rigid body modes), this gives

$$1 + \cos\lambda L \cosh\lambda L = 0$$

There are no closed-form solutions to this equation, so the roots $\lambda_r L$ must be obtained numerically and are given in the table on page 5. As before, the natural frequencies can be found using equation (5), which is the definition of the wavenumber, λ.

④ The **mode shapes** are obtained by substituting $\lambda = \lambda_r$ into equation (7) and solving for the constants C_1 - C_4.

From (7a) and (7b) $C_3 = -C_1$ and $C_4 = -C_2$

Thus from (7c) or (7d)

$$C_2 = -\frac{\sin\lambda_r L + \sinh\lambda_r L}{\cos\lambda_r L + \cosh\lambda_r L}\, C_1$$

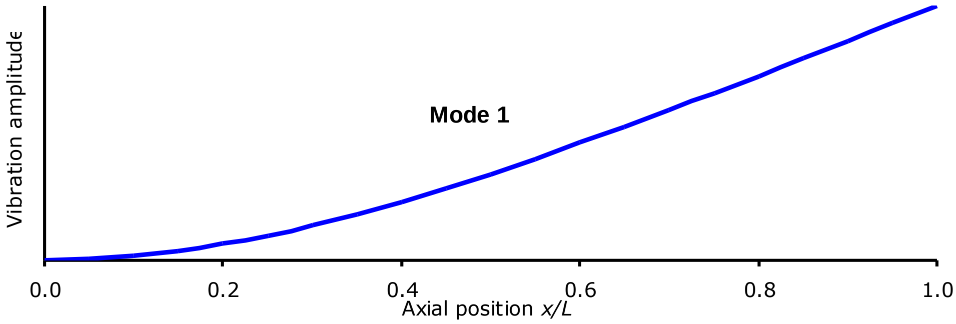
$$= \sigma_r\, C_1$$

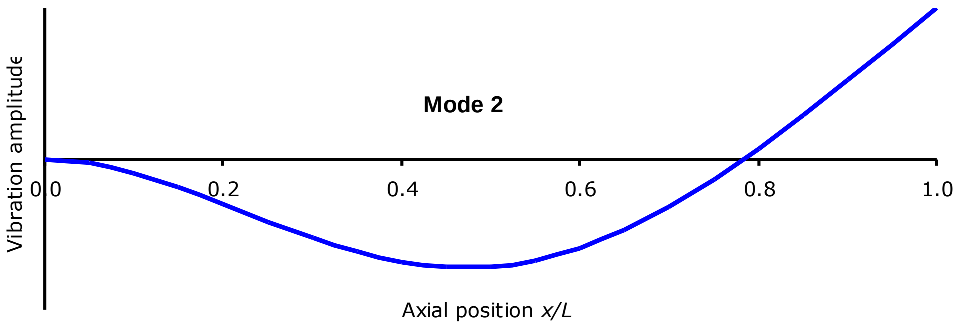
This gives C_2, C_3 and C_4 in terms of C_1, an arbitrary constant.

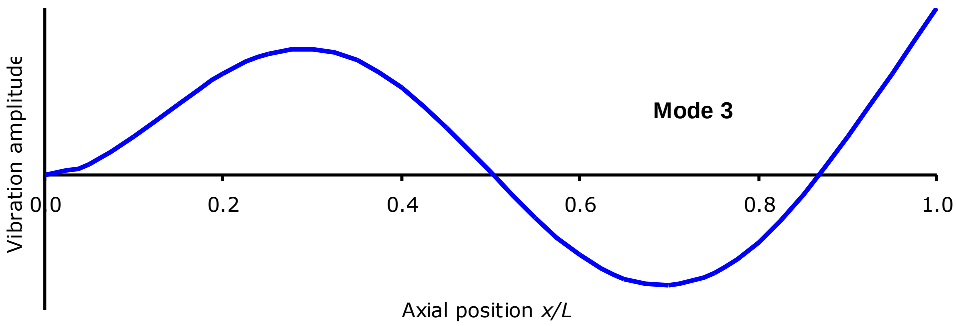
If we choose $C_1 = 1$, the mode shape becomes

$$Y_r(x) = \sin\lambda_r x - \sinh\lambda_r x + \sigma_r\left(\cos\lambda_r x - \cosh\lambda_r x\right)$$

When each value of λ_r is used in this equation, a different deflected shape is obtained.

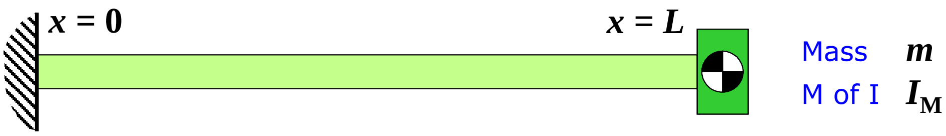






Other Boundary Conditions

Example   Cantilever Beam with a Mass at the Free End

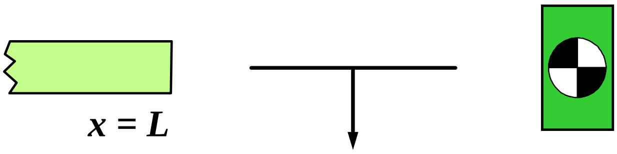


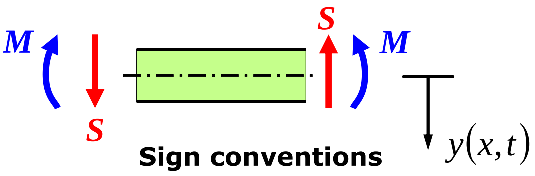
① The boundary conditions at the clamped end are identical to the previous case. So, $Y = 0$ and $\frac{dY}{dx} = 0$ at $x = 0$.

However, at $x = L$, $S \neq 0$ and $M \neq 0$. To look at the effect that the mass has on the vibration of the beam, we use two of the basic principles of Mechanics. These are

1. Compatibility of displacements
2. Equilibrium of forces and moments

Consider first the shear force reaction between the beam and the mass. The free body diagram is





Sign conventions
See Formula Sheet

Compatibility of displacements
Displacement at the end of the beam is the same as the displacement of the mass

Equilibrium of forces
Shear force on the beam is equal and opposite to the force on the mass

For the Beam

For the mass

Equating and noting that $\omega^2 = \frac{EI\lambda^4}{\rho A}$, $\left(\frac{d^3Y}{dx^3}\right)_{x=L} + \frac{m(\lambda L)^4}{\rho A L^4} Y(L) = 0$

Now consider the bending moment reaction between the beam and the mass.

Compatibility of displacements Slope at the end of the beam is the same as the rotation of the mass

Equilibrium of moments Bending moment on the beam is equal and opposite to the bending moment on the mass

For the Beam

For the mass

Slope, $\theta(t) = \dfrac{\partial y}{\partial x} = \left(\dfrac{\mathrm{d}Y}{\mathrm{d}x}\right)\cos\omega t$

Therefore, $\dfrac{\partial^2\theta}{\partial t^2} = -\omega^2\left(\dfrac{\mathrm{d}Y}{\mathrm{d}x}\right)\cos\omega t$

Equating, $\left(\dfrac{\mathrm{d}^2Y}{\mathrm{d}x^2}\right)_{x=L} - \dfrac{I_{\mathrm{M}}(\lambda L)^4}{\rho A L^4}\left(\dfrac{\mathrm{d}Y}{\mathrm{d}x}\right)_{x=L} = 0$

Collecting the four boundary condition equations together, we have

$$Y(0) = 0 \qquad \text{(a)}$$

$$\left(\frac{\mathrm{d}Y}{\mathrm{d}x}\right)_{x=0} = 0 \qquad \text{(b)}$$

$$\left(\frac{\mathrm{d}^3Y}{\mathrm{d}x^3}\right)_{x=L} + \frac{m(\lambda L)^4}{\rho A L^4}Y(L) = 0 \qquad \text{(c)}$$

$$\left(\frac{\mathrm{d}^2Y}{\mathrm{d}x^2}\right)_{x=L} - \frac{I_{\mathrm{M}}(\lambda L)^4}{\rho A L^4}\left(\frac{\mathrm{d}Y}{\mathrm{d}x}\right)_{x=L} = 0 \qquad \text{(d)}$$

② To set up equation (7) for this system, substitute for $Y(x)$ and its derivatives from equation (6).

Steps ③ and ④ follow as in the previous examples.